# Turbulent line vortices

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An attempt has been made to establish the laws governing the flow in a turbulent line vortex. Up to the present time theoretical solutions for laminar flow have been used for comparison with experimental results for turbulent flow to find an 'eddy viscosity' term and its variation with various parameters. An approach is developed along lines similar to the methods used in turbulent boundary-layer theory and is found to be reasonably successful as far as the work has proceeded. It is predicted by theory, and confirmed by experiment, that the circulation in the vortex is proportional to the logarithm of radius under certain conditions. For the present experimental conditions, the vortices are found to be completely independent of viscosity effects when the parameter $WZ/K_0$ exceeds 150, and above this value the experimental results may be correlated to give a universal distribution of circulation in the inner region of the vortex. Further experiments are necessary to verify and extend the results of these tests before any definite conclusions may be made regarding the circulation distribution in the outer core region of the vortex and the growth and development of the vortex.

## 1. Introduction

In recent years the study of the vortex has produced many interesting papers, concerned mainly with the line vortex as produced by an aircraft wing and also the driven vortex of the Ranque–Hilsche tube. However, these papers have been mainly concerned with laminar flow and as the flow in the above-mentioned applications would usually be turbulent, it was thought desirable to attempt a theoretical and experimental investigation of turbulent line vortices. In this paper the line vortex will be considered as this is possibly the more basic of the vortex types.

To the present time only one analysis has appeared for a line vortex with turbulent flow. This was due to Squire (1954). He considered the equations of motion for a fluid with both steady and fluctuating components of velocity and vorticity and obtained a solution which was similar in form to that for laminar flow except for the replacement of the kinematic viscosity by an eddy viscosity term.

Newman (1959) and Dosanjh, Gasparek & Eskinazi (1962) carried out experiments on turbulent line vortices. The process used in analysing experimental results has been to compare the vortex velocity or pressure distributions with the laminar flow theory of Newman's paper, the curves of this theory having been plotted for varying kinematic viscosity. By this method an 'effective viscosity' has been found for the vortex. However, it has been noted that the theoretical (laminar) and experimental (turbulent) curves did not correspond over the major portion of the profile which is sheared.

This discrepancy might well be explained by the fact that the mechanism of laminar and turbulent flows is entirely different—even to the extent that a turbulent flow may be completely independent of viscosity. This method gave solutions which may be satisfactory for some engineering applications.

## 2. Theoretical considerations

### (a) *Application of mixing-length theory to concentrically circular flow*

Of some success in the study of turbulent flows have been the mixing length theories of Prandtl, von Kármán and Taylor, of which the simplest is that of Prandtl. All these theories predict the logarithmic law for the turbulent boundary layer, which can also be regarded as a consequence of similarity arguments.

Townsend (1961) showed that the energy equation for a plane shear flow, when simplified, reduced to an equation which may be given a mixing-length interpretation. It is noted that for the case of a boundary layer, where little curvature of the flow exists, momentum is conserved in the mixing process. However, in the case of a vortex with concentrically circular flow, moment of momentum must be conserved.

Consider a fluid lump at radius $r$, having a mean radial velocity $u$, mean tangential velocity $v$, and fluctuating velocity components $u'$, $v'$. The Reynolds stress is given by

$$\tau = -\rho\overline{u'v'} = -\rho\beta(\overline{u'^2})^{\frac{1}{2}}(\overline{v'^2})^{\frac{1}{2}},$$

where $\beta$ is a correlation coefficient defined by

$$\beta = \overline{u'v'}/(\overline{u'^2})^{\frac{1}{2}}(\overline{v'^2})^{\frac{1}{2}}.$$

Then $\tau = \{-\rho(\overline{v'^2})^{\frac{1}{2}}r\}\{\beta(\overline{u'^2})^{\frac{1}{2}}/r\}$. The quantity $-\rho(\overline{v'^2})^{\frac{1}{2}}r$ is a fluctuating component of moment of momentum. Now assume that the element leaving the layer has the mean moment of momentum of that layer (i.e. $-\rho vr$) and this moment of momentum is conserved over a mixing length $l$. The gradient of moment of momentum is assumed constant over the small distance $l$. Then

$$-\rho(\overline{v'^2})^{\frac{1}{2}}r = l\,d(-\rho vr)/dr.$$

Hence

$$\tau = -l\rho(\overline{u'^2})^{\frac{1}{2}}\beta\left(\frac{dv}{dr}+\frac{v}{r}\right),$$

or

$$\tau/\rho = \nu_T\xi, \tag{1}$$

where $\nu_T$ is an eddy viscosity and $\xi = (dv/dr+v/r)$ is the vorticity. An expression similar to this was given by Prandtl (1929).

### (b) *Solution of equation*

In simple theories of free turbulent flows it is usually assumed that the eddy viscosity is constant. In the present case this may be shown to be consistent with a more detailed argument, based on the assumption that the eddy viscosity depends only on the local shear stress $\tau$, density $\rho$ and radial position $r$. Dimensional analysis then leads to

$$\nu_T = \mathscr{K}(\tau/\rho)^{\frac{1}{2}} r, \tag{2}$$

where $\mathscr{K}$ is a constant. If the 'inertia' terms in the equation of motion are neglected as being small compared with the Reynolds stresses, the equation of motion for the rotational flow reduces to

$$\partial(r^2\overline{u'v'})/\partial r = 0,$$

or

$$\tau/\rho = -\overline{u'v'} = A/r^2, \tag{3}$$

where $A$ = const. Combining (2) and (3) shows that

$$\nu_T = \mathscr{K}A^{\frac{1}{2}} = \text{const.} \tag{4}$$

Substituting (4) into (1) and using $\xi = 1/r\,(\partial K/\partial r)$, where $K$ is the circulation, we then have

$$\partial K/\partial r = A^{\frac{1}{2}}/\mathscr{K}r,$$

or

$$K = A^{\frac{1}{2}}\,\mathscr{K}^{-1}\ln r + \text{const.} \tag{5}$$

Now

$$A = (\tau/\rho)\,r^2 = \text{const.},$$

giving $A^{\frac{1}{2}} = v_\tau r = K_\tau = \text{const.}$ $K_\tau$ may be called the 'friction circulation' by analogy with the friction velocity as used in boundary layers. Note that $K_\tau$ is a local value and is constant for the logarithmic region of any given vortex. Equation (5) becomes

$$K/K_\tau = \mathscr{K}^{-1}\ln r + \text{const.} \tag{6}$$

Hence if the assumed mechanism is approximately valid for this flow, the circulation distribution will be logarithmic in regions where the inertia forces are negligible compared to the Reynolds stresses. A similar expression is obtained by using a modified form of von Kármán's similarity hypothesis. Experiments have shown (6) to be correct.

From (6) several interesting deductions may be made regarding the velocity and circulation distributions. In the logarithmic region of circulation, the velocity and circulation are given by

$$v \propto r^{-1}\ln r, \quad K \propto \ln r,$$

which gives

$$r \to 0, \quad v \to -\infty$$

$$r \to \infty, \quad K \to \infty.$$

As both of these are impossible it is obvious that there is a change of mechanism of the flow at radii both less and greater than the logarithmic region so that we have $v = 0$ at $r = 0$, and $K = K_0$ at $r = \infty$ for the free field.

In the centre of the vortex there will be a region of solid body rotation so that $K \propto r^2$. This is the 'eye' of the vortex in which the shear stresses are small as there is little slip between concentric layers of fluid. In this region, and a small region outside it, tangential inertia forces may be expected to dominate since there may be rapidly changing tangential velocities as well as changing radial velocities due to wake velocity defects.

In the region of the tangential velocity peak, however, the tangential inertia forces are small but the shear stresses are large, so that it is in this region that the logarithmic distribution of circulation is to be expected.

Between the region of solid rotation and the logarithmic region may be expected some form of transition curve similar to the 'buffer' region in turbulent boundary layers. However, there are differences between the two cases; with the vortex the shear stress is zero at the origin, whereas it is often a maximum at this position for a boundary layer. Another difference is the necessity of a viscous sublayer in a boundary layer due to the damping of the turbulent velocity fluctuations close to the wall. The vortex is a free flow without boundary restrictions and fluctuations are not necessarily damped in the region of the origin, even though the shear stress is zero due to the solid body rotation.

If the inner region of the vortex is correlated by use of equation (6) then the outer-core region may vary with other flow parameters, as is the case with the turbulent boundary layer.

### (c) *Alternative solution by dimensional reasoning*

The authors are indebted to Mr A. E. Perry of the Mechanical Engineering Department for the following dimensional reasoning, which also leads to equation (6).

It is found from experiment that by plotting $v/v_1$ versus $r/r_1$, where $v_1$ and $r_1$ refer to a characteristic point in the velocity distribution (such as a point of maximum tangential velocity) all points for the inner region fall on one curve quite independently of the outer flow characteristics in which no simple similarity law was found. This indicates that no more than five variables are involved in the inner region and these would most likely be

$$v = f(r, r_1, \rho, \tau_1),$$

giving

$$\frac{v}{v_{\tau_1}} = f\left(\frac{r}{r_1}\right) \quad \text{or} \quad \frac{K}{K_{\tau_1}} = f\left(\frac{r}{r_1}\right).$$

The above equation might be called the 'law of the vortex core' by analogy with Prandtl's law of the wall with turbulent boundary layers.

Near the centre of the vortex, radial outflow and high tangential velocity gradients were observed and the momentum equation therefore indicates high tangential inertia in this region. Further away from this central region these effects become small and so a considerable outer region of low inertia and hence constant shear moment or constant $K_\tau$ would exist. The radius of this central region of high inertia would be proportional to $r_1$ since a law of the core is applicable.

The size of the core or the processes occurring inside of it should not influence the distribution of vorticity in the outer flow since the mechanisms are seen to be quite different in the two regions. Therefore a variable such as $\partial K/\partial r$ ($= r\xi$) in the outer flow should not depend on $r_1$ but would still depend on $K_\tau$ (i.e. the shear moment being transmitted) and perhaps other variables. Since a sudden discontinuity would not be expected between the outer flow and the region in which the law of the core is applicable, a small but finite blending region should exist where both conditions are satisfied, that is, from the law of the core,

$$\frac{\partial}{\partial r}\left(\frac{K}{K_\tau}\right) = \frac{1}{r_1} f'\left(\frac{r}{r_1}\right) + \text{function involving } r_1,$$

and hence $f'$ must be given by $\mathscr{H}^{-1}(r_1/r)$, where $\mathscr{H}$ is a constant. Hence

$$\frac{\partial}{\partial r}\left(\frac{K}{K_\tau}\right) = \frac{1}{\mathscr{H} r}$$

which has equation (5) as a solution, and since $K/K_\tau = f(r/r_1)$ then

$$\frac{K}{K_\tau} = \frac{1}{\mathscr{H}} \ln \frac{r}{r_1} + \text{const.},$$

which is similar to equation (6). The same result is obtained no matter what derivative of the circulation profile is taken as representing the shape of the region with $r > r_1$.

The above argument shows that a logarithmic law is at least possible from plausible physical interpretations of the mean velocity profile rather than referring to detailed models of the turbulence mechanism or simply 'curve fitting' the results. The result implies a moment of momentum transfer theory but is arrived at in a different way.

Note that the above reasoning would imply a defect law of circulation, but not necessarily one which is universal in nature. What is implied by the above analysis, however, is that there is a universal inner region.

### (*d*) *Local equilibrium of the flow*

Townsend (1961) defines an equilibrium layer as 'one in which there is equilibrium existing between the local rates of turbulent energy production and dissipation. The local rates of energy production and dissipation are so large, compared with the magnitude of the other terms in the energy equation, that the turbulent motion in this region is determined by the distribution of shear stress alone and is independent of conditions outside of the region.'

It is intended to show that the logarithmic region of the turbulent vortex may be such a region of local equilibrium.

The energy equation for the turbulent velocity components in the rotational direction is given by Traugott (1958) as

$$\frac{1}{2}\left[u\frac{\partial \overline{v'^2}}{\partial r} + w\frac{\partial \overline{v'^2}}{\partial z}\right] + \left[\overline{u'v'}\frac{1}{r}\frac{\partial}{\partial r}(vr) + \overline{v'w'}\frac{\partial v}{\partial z} + \overline{v'^2}\frac{u}{r}\right]$$
$$= -\frac{1}{\rho}\frac{\overline{v'}}{r}\frac{\overline{\partial p'}}{\partial \theta} - \frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}\overline{ru'v'^2} + \frac{\partial}{\partial z}\overline{w'v'^2} + \frac{2}{r}\overline{u'v'^2}\right]$$
$$-\nu\left[\overline{\left(\frac{\partial v'}{\partial r}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z}\right)^2} + \overline{\left(\frac{1}{r}\frac{\partial v'}{\partial \theta}\right)^2} + \frac{\overline{v'^2}}{r^2} - \frac{2}{r}\overline{v'\frac{1}{r}\frac{\partial u'}{\partial \theta}}\right],$$

where

(i) the first term is the convective rate of change of $\overline{v'^2}$ turbulent energy,

(ii) the second term is the production of turbulent energy from the mean velocity gradient (a positive production is associated with a negative value of this term),

(iii) the third term is the change in the turbulent energy due to the fluctuating pressure-velocity correlation or the work done against the fluctuating pressure gradients,

(iv) the fourth term is the change due to turbulent diffusion,

(v) the fifth term is the change due to the action of viscosity (viscous dissipation).

Now if the flow has local equilibrium according to the conditions prescribed by Townsend, and dropping terms of small magnitude, the equation reduces to

$$-\overline{u'v'}\frac{1}{r}\frac{\partial}{\partial r}(vr) = \epsilon,$$

where $\epsilon$ is the local rate of conversion of turbulent energy to heat.

By use of flow similarity arguments as in Townsend (1961) we write

$$\epsilon = (\overline{q^2})^{\frac{3}{2}} L_\epsilon^{-1} \quad \text{and} \quad \overline{u'v'} = a_1(\overline{q^2}),$$

where $$\overline{q^2} = \overline{u'^2}+\overline{v'^2}+\overline{w'^2}.$$

These combined with the above equation give

$$\frac{l}{r}\frac{\partial K}{\partial r} = -(\overline{u'v'})^{\frac{1}{2}},$$

where $l = a_1^{\frac{3}{2}} L_\epsilon$ may be interpreted as a form of 'mixing length'. With

$$\tau/\rho = -\overline{u'v'} = A/r^2,$$

the equation is $$l\partial K/\partial r = A^{\frac{1}{2}} = \text{const.}$$

If now we assume $l \propto r$ equation (5) is obtained.

In other turbulent free flows, the mixing length is assumed constant at any cross-section and proportional to the width of the flow. In such flows streamlines are usually nearly parallel and velocity gradients small so that there is no characteristic length scale associated with the flow which is a function of position. The assumption of $l =$ const. then gives results in reasonable agreement with experiment.

However, in the present case, the streamlines are circular and hence the flow processes are likely to be largely dependent on the local curvature, especially near the vortex centre where the flow curvature varies rapidly. For this reason it may be justifiable to assume that $l = l\,(r, \ldots)$ and since no other characteristic length variables appear to be involved we may write $l = \mathscr{H}r$, where $\mathscr{H}$ is a constant. If this is so, the region described by equation (5) is possibly an equilibrium layer.

### (e) *Choice of reference circulation and radius*

The previous analyses show that a reference 'friction circulation' $K_\tau$ must be used in correlating results. However, as yet no method for the determination of $K_\tau$ has been devised and so some other reference is required.

The obvious choice of a reference radius for correlation of the inner region would appear to be the radius ($r_1$) at which the maximum tangential velocity occurs as this is the only distinctive radius for this region of the vortex. If this radius is used, then the correlating circulation must be that at radius $r_1$. This circulation shall be called $K_1$. However, it cannot be readily said that $K_1$ applies throughout the flow as does $K_\tau$.

Dimensional reasoning gives

$$K_1 = f(r_1, \rho, \tau)$$

since these are the only independent variables involved. This gives

$$K_1/K_\tau = \text{const.},$$

so that equation (6) becomes

$$\frac{K}{K_1} = \frac{1}{\mathscr{H}} \ln\left(\frac{r}{r_1}\right) + 1. \tag{7}$$

The above reference values may not necessarily be the correct ones, but the same reasoning will apply to any values chosen within the inner region. It may be possible to set up an 'angular momentum integral' method for determination of $K_\tau$, similar to methods used in turbulent boundary layer work.

### (*f*) *Dimensional analysis and design of experiments*

As dimensional analysis has proved so useful in the analysis of other turbulent flows it might also be expected to be helpful in this case, especially for the designing of experiments on turbulent vortices.

The main determining feature of a vortex is its circulation distribution and the external variables likely to affect the flow in the vortex are density, viscosity, free-field circulation, free-stream velocity, distance downstream from the point of generation of the vortex and radius from the axis of symmetry. Hence

$$f_1(K, K_0, W, r, Z, \mu, \rho) = 0,$$

or

$$\frac{K}{K_0} = f_2\left(\frac{WZ}{K_0}, \frac{r}{Z}, \frac{WZ}{\nu}\right), \tag{8}$$

which is a four parameter family.

If the vortices are turbulent and fully developed then it might be expected that the flow is independent of viscosity so that equation (8) becomes

$$\frac{K}{K_0} = f_2\left(\frac{WZ}{K_0}, \frac{r}{Z}\right). \tag{9}$$

To determine whether the flow has any dependency on viscosity is simply a matter of performing several experimental runs at a constant value of $WZ/K_0$, but widely varying values of $WZ/\nu$. If the plots of these results are similar (when reduced to suitable dimensionless form), the flow may be said to be independent of viscosity.

An alternative, and simpler, method to show the independence of viscous effects, is to perform experiments over as wide a range of $WZ/K_0$ as possible and apply the following reasoning to the experimental results. In correlating the results of experiments, $K_\tau$ may not be known, and so it may be necessary to use some circulation which is proportional to, or is a function only of, $K_\tau$. In using this correlating circulation $K_1$ and the corresponding radius $r_1$ in the equation

$$\frac{K}{K_1} = \frac{1}{\mathscr{H}} \ln\left(\frac{r}{r_1}\right) + 1,$$

it will first be necessary to show that $K_1/K_0$ and $r_1/Z$ are independent of the effects of viscosity, this showing the independence of the inner region from viscous effects.

For a distributive dependent variable equation (8) holds, and for a non-distributive dependent variable we have

$$\frac{K_1}{K_0} = f\left(\frac{WZ}{K_0}, \frac{K_0}{\nu}\right), \quad \frac{r_1}{Z} = f\left(\frac{WZ}{K_0}, \frac{K_0}{\nu}\right).$$

If equation (9) is to apply to the experimental results, i.e. $K_0/\nu$ is not involved in $K/K_0$, then it can be reasoned that

$$\frac{K_1}{K_0} = \text{either} \quad f\left(\frac{WZ}{K_0}\right) \quad \text{or} \quad f\left(\frac{K_0}{\nu}\right)$$

and

$$\frac{r_1}{Z} = \text{either} \quad f\left(\frac{WZ}{K_0}\right) \quad \text{or} \quad f\left(\frac{K_0}{\nu}\right).$$

Now, when $r_1$ and $K_1$ are used as independent variables equation (9) becomes

$$\frac{K}{K_1} = f\left(\frac{r}{r_1}, \frac{WZ}{K_1}\right), \tag{10}$$

which is a more suitable form for correlating experimental results.

It is necessary to plot values of $K_1/K_0$ versus $WZ/K_0$ and if one curve is produced for various values of $K_0/\nu$ then $\nu$ is not involved in the inner region. A similar procedure is followed for $r_1/Z$. Hence the flows which are completely independent of viscosity may be found.

## 3. The experimental investigation

The aims were:

(1) to generate turbulent line vortices in a wind tunnel and to determine under what conditions the flow is completely independent of viscosity;

(2) to determine whether the logarithmic region of circulation as predicted by theory is obtained in practice;

(3) to attempt to correlate results in the vortex inner region as is done for turbulent boundary layers.

### (a) *Apparatus and experimental method*

The low-speed wind tunnel of the Royal Melbourne Institute of Technology was used for the experimental programme. The turbulence level is 0·23% at 65 ft./sec. The working section is parallel with approximately zero pressure gradient.

A yaw probe similar to that described by Templin (1954) was used in conjunction with a traversing gear situated over the tunnel roof. Both were remotely controlled, the accuracies being $\pm 0{\cdot}1^\circ$ in rotation, and $\pm 0{\cdot}001$ in. in traverse. The yaw tube consisted of three 1 mm tubes soldered together, the two outer tubes being cut at an included angle of 80°. For further details see Hoffmann (1962).

A 'differential' aerofoil was used to generate vortices. This consisted of a 6 in. chord wing spanning the tunnel vertically, the lower half being mounted at an angle of incidence equal and opposite to that of the upper half. The advantage of this arrangement is that a stable single vortex is produced, whose position in the tunnel remained almost completely independent of velocity, angle of attack and distance downstream.

The experiments were carried out for $WZ/K_0$ values ranging from 64 to 511, these values being determined by the limitations of test-section length and the distance from the wing required for the vortex to be reasonably well formed.

The range of experimental conditions are shown in table 1 below.†

| Run no. | $K_0$ ft.²/sec | $K_1$ ft.²/sec | $W$ ft./sec | $Z$ in. | $r_1$ in. | $r_0$ in. | $\frac{K_1}{K_0}$ | $\frac{r_1}{r_0}$ | $\frac{WZ}{K_0}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6·46 | 2·08 | 82·7 | 59 | 0·485 | 12·0 | 0·322 | 0·0404 | 63·5 |
| 2 | 2·085 | 0·785 | 37·35 | 59 | 0·47 | 8·3 | 0·376 | 0·0567 | 88·3 |
| 3 | 5·75 | 2·34 | 105·6 | 59 | 0·48 | 8·5 | 0·407 | 0·0565 | 89·7 |
| 4 | 2·81 | 1·01 | 61·0 | 49·75 | 0·46 | 7·4 | 0·36 | 0·0622 | 90 |
| 5 | 2·14 | 0·922 | 57·2 | 59 | 0·43 | 6·7 | 0·414 | 0·0642 | 111·5 |
| 6 | 3·25 | 0·875 | 79·8 | 59 | 0·415 | 8·3 | 0·269 | 0·050 | 121 |
| 7 | 2·67 | 0·738 | 81·7 | 59 | 0·52 | 7·2 | 0·277 | 0·0722 | 151 |
| 8 | 1·54 | 0·588 | 82·0 | 59 | 0·63 | 4·8 | 0·389 | 0·131 | 261 |
| 9 | 1·083 | 0·483 | 76·7 | 59 | 0·72 | 3·7 | 0·446 | 0·194 | 348 |
| 10 | 0·758 | 0·418 | 78·0 | 59 | 0·85 | 2·6 | 0·552 | 0·327 | 511 |

TABLE 1. Values of the experimental variables. $K$ denotes velocity multiplied by radius

Two corrections have been applied to these results: (i) a correction for the roll angles of the yaw tube when in a velocity gradient; (ii) a displacement correction for position of the yaw tube from the vortex centre.

### (*b*) *Semi-logarithmic plots of circulation distribution*

It is predicted by equation (6) that the distribution of circulation is proportional to log-radius if the inertia terms in the equation of motion are negligible compared to the Reynolds stresses. When the corrected experimental results of circulation are plotted on semi-logarithmic graph paper it is found that all profiles have a straight-line region which occurs near and slightly beyond the point of maximum tangential velocity. A typical profile of circulation is shown in figure 1. The experimental results showed two basic types of circulation distribution. This may be best seen by consideration of figure 2, where for some runs the extension of the log-line will cut the circulation line for solid body rotation (i.e. $K \propto r^2$), whereas for other runs no intersection will occur. This effect has not been fully investigated, but it is almost certainly due to the effects of viscosity, as will be shown in a later section.

### (*c*) *Independence of viscosity*

The dimensional reasoning of § 2 (*f*) shows that the condition for complete independence of viscosity in the inner region is that $K_1/K_0$ when plotted against $WZ/K_0$ for various values of $WZ/\nu$ should produce a single curve if $\nu$ is not to

† Complete experimental results are being held by the Editor. Those interested may obtain them upon request.

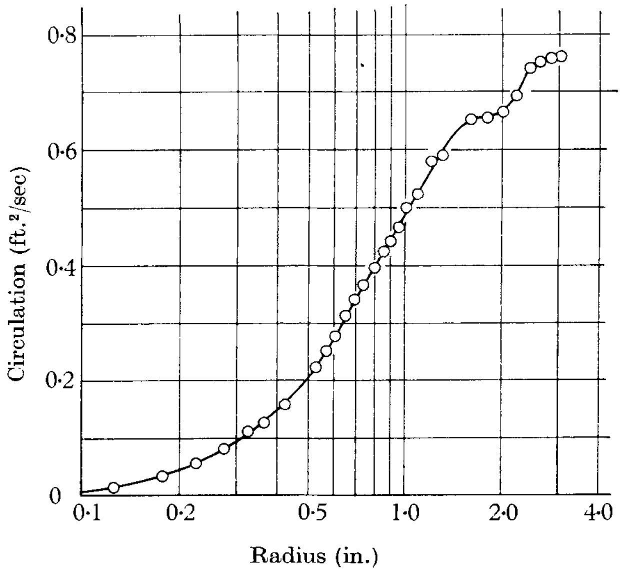


FIGURE 1. Typical experimental circulation profile showing region of circulation proportional to log-radius ($WZ/K_0 = 511$).

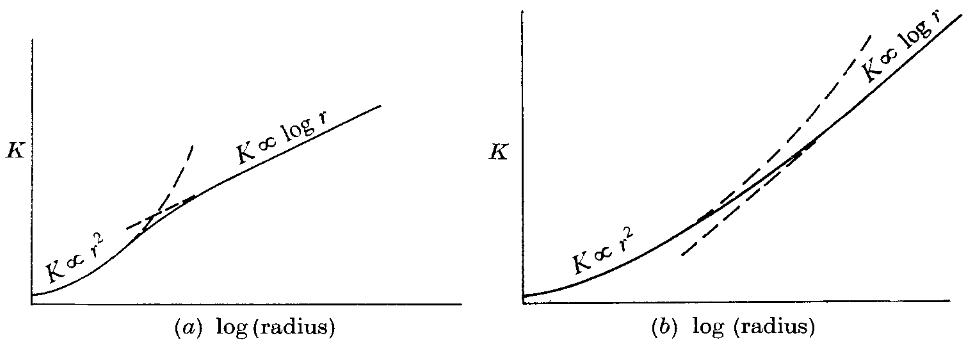


FIGURE 2. The two basic shapes of the circulation profile in the inner region obtained in the experiments.

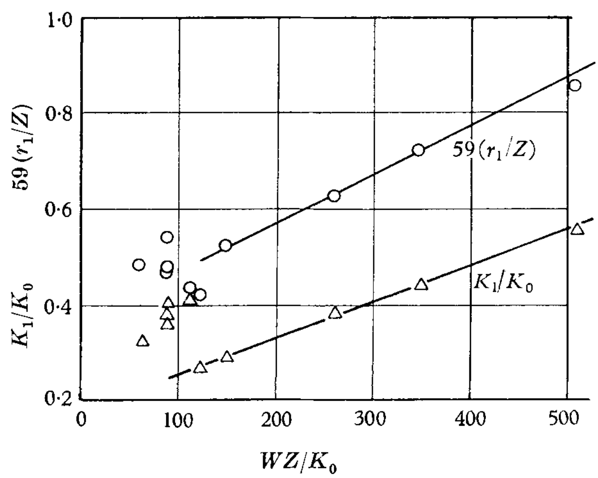


FIGURE 3. Graph of $K_1/K_0$ and 59 $(r_1/Z)$ versus $WZ/K_0$ used to determine those profiles which are completely independent of the effects of viscosity.

be involved, and similarly for $r_1/Z$ versus $WZ/K_0$. Hence the runs which are completely independent of viscosity (and so should produce universal distributions of circulation in the inner region) may be found.

In figure 3 the above parameters are plotted and it is seen that four runs (and a fifth nearly) lie on a straight line. The remaining points are scattered. Hence it is to be expected that these five runs will be close to universal in the inner region when plotted as the parameters $K/K_1$ versus $r/r_1$.

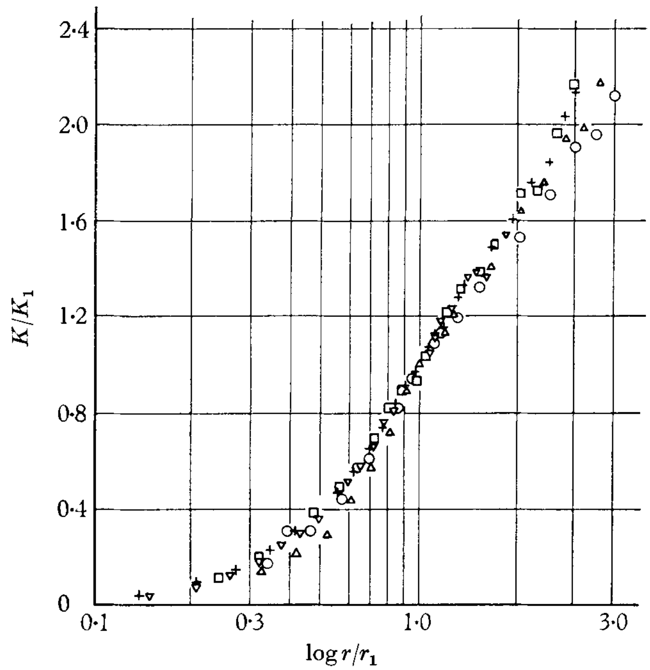


FIGURE 4. The universal circulation distribution of the vortex inner region.

| Symbol | Run no. | $WZ/K_0$ |
|---|---|---|
| ○ | 6 | 121 |
| △ | 7 | 151 |
| □ | 8 | 261 |
| + | 9 | 348 |
| ▽ | 10 | 511 |

Although the above test does not appear to be very rigorous in that there is a possibility that the points through which the line is drawn may not actually belong to the same curve, it is nevertheless encouraging that the circulation distributions corresponding to the above five points are close to being universal in the inner region, as shown in figure 4.

The condition for complete independence of the effects of viscosity appears to be $WZ/K_0 > 150$. This is unusual since it would be expected that the criterion should include a viscosity term. However, the viscous parameters $WZ/\nu$ and $K_0/\nu$ do not appear to yield any criterion. It is possible that more detailed experimental results are required before any conclusion may be reached regarding the condition for complete independence of viscosity.

The quoted value of $WZ/K_0 = 150$ may only apply to the present experimental conditions, since different methods of vortex generation would produce a different initial vortex. If the growth line of figure 3 is extrapolated to $r_1/Z = 0$, this point may be taken as the virtual origin for the production of the vortex. A criterion for $WZ/K_0$ based on this virtual origin may be more realistic since it eliminates effects due to the method of generation.

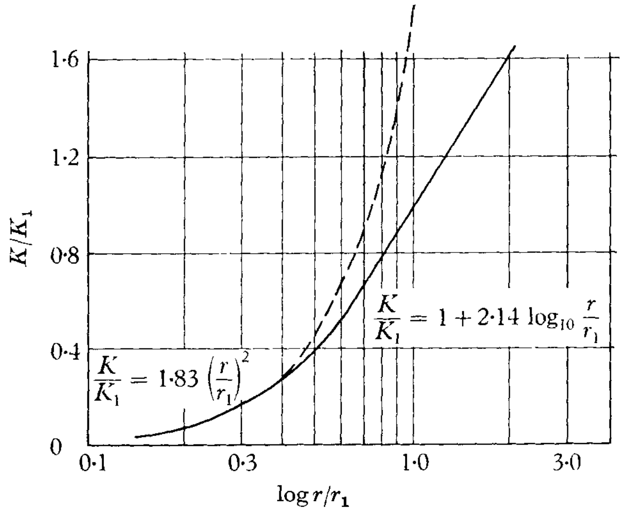


FIGURE 5. The universal circulation distribution for fully turbulent vortices and approximate numerical values for the constants.

### (d) *Circulation and velocity profile of the universal inner region*

Referring to figure 5, which shows the universal inner region for fully turbulent vortices, it is seen that there are three parts into which this region may be separated. These are

(i) an 'eye' of solid body rotation given by

$$K/K_1 = 1{\cdot}83(r/r_1)^2; \qquad (11)$$

(ii) a transition between the solid body rotation and the logarithmic circulation;

(iii) a region in which circulation varies logarithmically with radius and is given by

$$K/K_1 = 2{\cdot}14\log_{10}(r/r_1)+1. \qquad (12)$$

From the faired curve of the universal circulation plot it is possible to calculate the dimensionless velocity distribution for a fully turbulent vortex, since

$$\frac{K}{K_1} = \frac{v}{v_1}\frac{r}{r_1} = f\left(\frac{r}{r_1}\right).$$

The resulting velocity distribution for the fully turbulent vortex is shown in figure 6 compared with the distribution for a laminar vortex, using the same co-ordinates.

In figure 3, the curve connecting the points giving a universal distribution of circulation is, within experimental accuracy, close to a straight line. The five points considered were at constant distance downstream, so that these results imply that the size of the vortex eye, for given free-stream conditions, grows linearly with distance downstream from the origin of generation. Note that this behaviour is similar to that of laminar and turbulent jets. A laminar jet grows parabolically with distance (as does a laminar vortex) and a turbulent jet grows linearly with distance downstream.

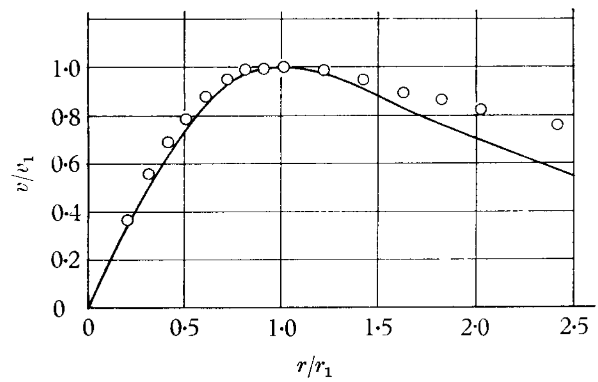


FIGURE 6. Comparison of laminar and turbulent velocity profiles in the inner region. ——, Laminar; ○, turbulent.

Hence if the free-stream conditions are known, the inner region of a fully developed turbulent vortex is predicted from figures 3 and 5, as may be done for a turbulent boundary layer. However, this may not be so for another method of generation, due to different conditions at generation.

### (*e*) *Outer core region*

It was stated in § 2 (*c*) that the dimensional reasoning used to obtain a universal inner 'law of the core' implied that a circulation defect law would exist in the outer core region, though this need not necessarily be universal in nature. If the analogy between boundary layers and vortices is extended further, it might be expected that the form of the defect law would be

$$(K_0 - K)/K_\tau = f(r/r_0),$$

or, since in fully turbulent vortices it is shown that $K_1 \propto K_\tau$,

$$(K_0 - K)/K_1 = f(r/r_0), \tag{13}$$

where $r_0$ is the radius at which the circulation $K_0$ is achieved (or radius for 99 % $K_0$, as this is more easily defined).

There is difficulty in verifying this defect law due to the poor accuracy of measurements of circulation at large radii and consequent errors in the calculations. Also in this outer region there is an unexplained discontinuity which appears in the circulation distribution. In this area the circulation remains approximately constant. At the present it is not known for certain whether this bump is a property of the turbulent vortex or is due to incomplete rolling up of the trailing vortex at the plane of measurement or to some other cause. It may be possible that the flow has not developed sufficiently for a universal defect law to be formed. Due to lack of accurate experimental results and the above-mentioned difficulties, the only attempt to find a defect law is that shown in figure 7 plotted on the basis of equation (13). The two curves plotted are those for which the irregularity on the circulation distribution is the smallest.

Further experimental work and closer investigation of the irregularity in the circulation distribution is necessary before any conclusion may be drawn regarding a circulation defect law.

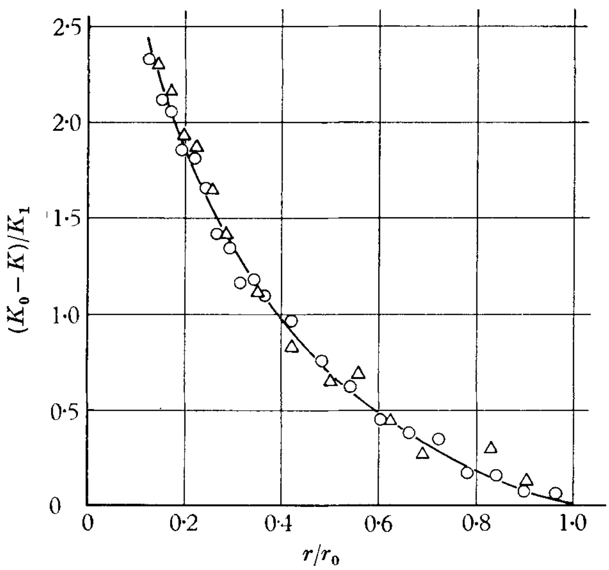


FIGURE 7. A possible circulation defect law plotted for two runs only. ⊙, Run no. 6; △, run no. 7.

### (*f*) *Development of a fully turbulent vortex*

Earlier it was noted that over the fully turbulent range of the parameter $WZ/K_0$, the eye of the vortex grows linearly with distance downstream from the point of generation. It is interesting to investigate this linear growth further and see what limitations are placed on it. This is best done by plotting $K_1/K_0$ and $r_1/r_0$ as a function of $WZ/K_0$ for the various fully turbulent vortices, as in figure 8.

As the vortex strength $K_0$ will decay very slowly with distance downstream, it is seen that as the vortex progresses both $K_1/K_0$ and $r_1/r_0$ increase with distance downstream. This means that as $Z$ increases, so the radius at which maximum tangential velocity is realized approaches closer to the radius at which the free-stream circulation $K_0$ is obtained.

It is tempting to extrapolate these curves and investigate their implications. If the linear growth curve is extrapolated to $r_1/r_0 = 1$ along the path A, it is at once obvious that at this point $K_1/K_0$ must also be equal to unity and the vortex must be of the Rankine type.

However, it is highly unlikely that this type of vortex could be produced in practice. With this extrapolation, if we continue past $K_1/K_0 = 1{\cdot}0$, the vortex now has two radii at which the free-stream circulation, exists and hence it is unstable by Rayleigh's criterion for the stability of a rotational flow (Rayleigh 1916). It is unlikely therefore that this path of growth is continued for values of $WZ/K_0$ much larger than the experimental upper value of 511. However, if this did occur, it might give rise to a possible explanation of vortex instability.

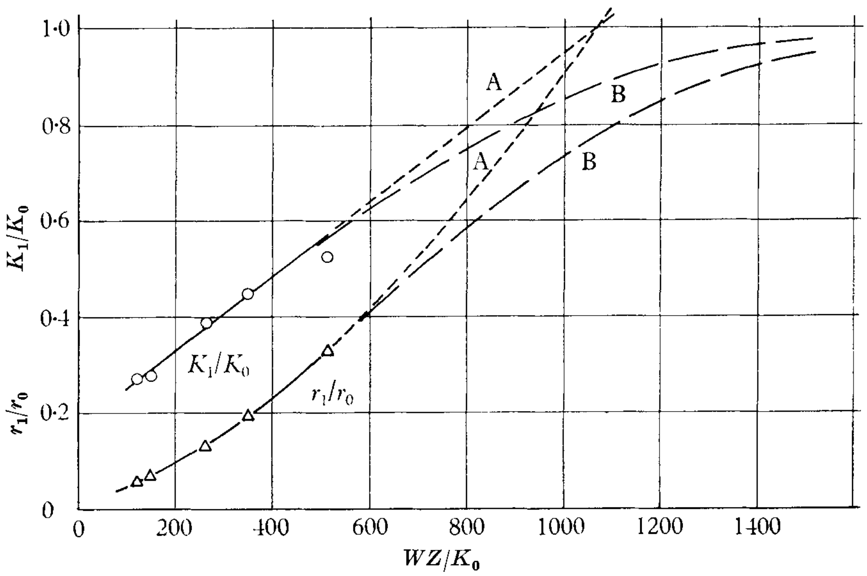


FIGURE 8. Growth of a fully turbulent vortex and possible extrapolations to higher values of $WZ/K_0$.

The second, and more likely, growth curve is the path B shown on figure 8. Here $r_1/r_0$ and $K_1/K_0$ approach unity asymptotically for large values of $WZ/K_0$.

Although the foregoing paragraphs are largely conjectured (since the results are extrapolated), the remarks may provide an avenue for further research into growth and stability of turbulent vortices. Obviously further experiments at high values of $WZ/K_0$ are required before definite conclusions may be drawn.

### (*g*) *Comparison with other experimental results*

In figure 9 are plotted the circulation distribution of vortices from Newman (1959) and Timme (1957). The vortex from Newman's report was produced by a single wing and has a $WZ/K_0$ value of approximately 290 and so is expected to be fully turbulent. It is seen that there is reasonable correspondence between this profile and those of the present work. As the original results were not available, both profiles are scaled from the figures in the references quoted.

Timme's vortex is a single vortex of a Kármán vortex street and was taken from his figure 19. The circulation distribution was calculated by taking the origin at the centre of vorticity rather than the point of zero velocity. Once again reasonable correlation is obtained. Although this vortex is fully turbulent

(Timme reports an eddy viscosity of approximately $10\nu$) the value of $WZ/K_0$ is approximately 0·3. Apparently a different criterion exists for independence of viscosity in the Kármán vortex street. $Z$ in this instance was taken to be the product of the free-stream velocity and the time from generation of the vortex.

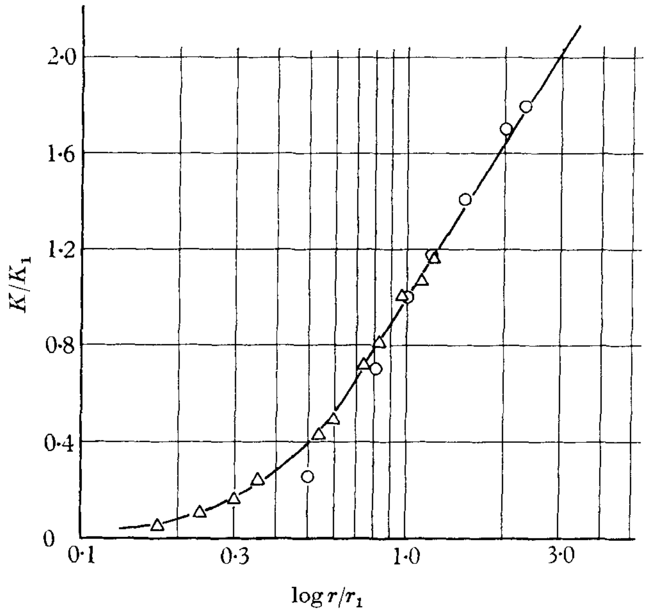


FIGURE 9. Comparison of the present experimental results with those of Newman (1959) and Timme (1957). ○, Newman; △, Timme (vortex street).

## 4. Possibilities for future experimental work

At several instances throughout this work, the necessity for further experimental work has been noted. As a guide a useful programme would be:

(1) accurate measurements in the outer core region and a search for a possible circulation defect law;

(2) experimental results are required at lower values of $WZ/K_0$ than in the present experiments to investigate the manner in which viscosity enters the problem;

(3) tests at values of $WZ/K_0$ greater than 500 are needed to study the behaviour of the vortex as it approaches the Rankine type and to see whether instability occurs near this condition;

(4) more detailed results are needed in the present experimental range to verify the conclusions drawn in the present work;

(5) development of a means for determining the friction circulation $K_\tau$.

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